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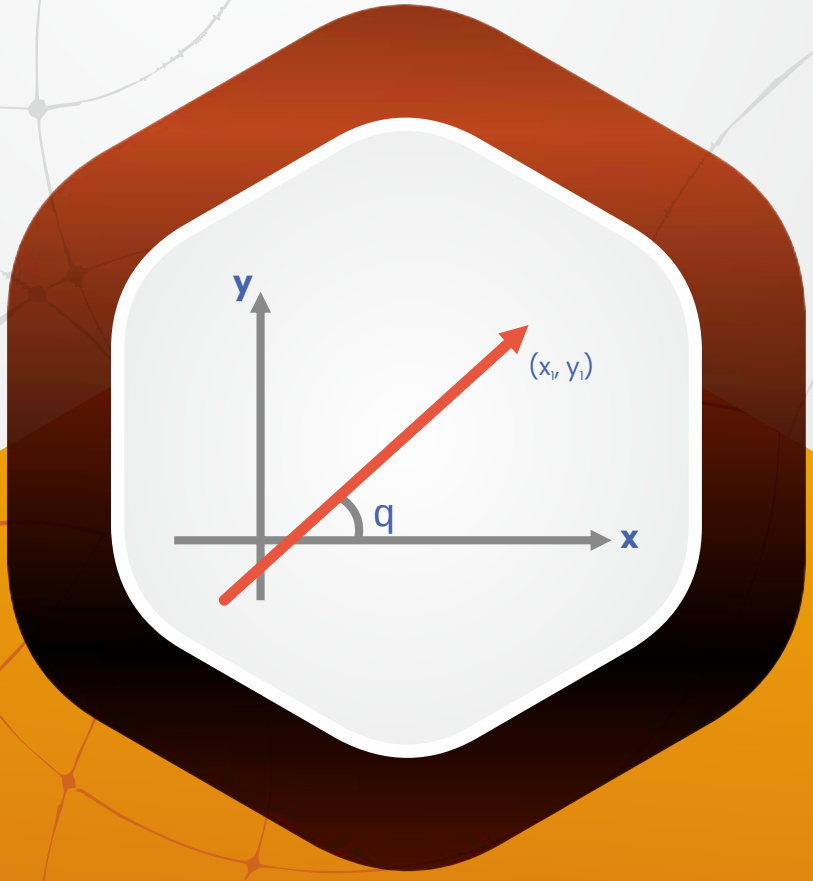
MATHEMATICS

JEE (MAIN+ADVANCED)

NURTURE COURSE



Study Material

Straight Line

(English Medium)

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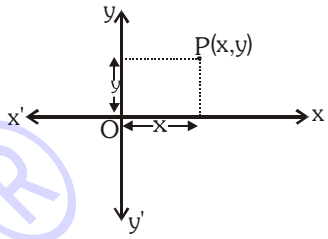
POINT & STRAIGHT LINE

1. INTRODUCTION OF COORDINATE GEOMETRY :

Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

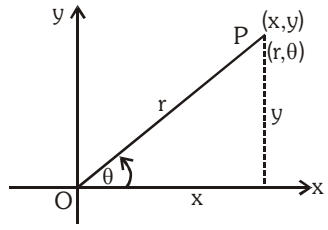
2. CARTESIAN CO-ORDINATES SYSTEM :



In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative.

Distances along the y-axis above the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). The y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

3. POLAR CO-ORDINATES SYSTEM :



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

4. DISTANCE FORMULA AND ITS APPLICATIONS :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note :

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

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(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if AB = BC = CD = DA & AC = BD ; $AC \perp BD$

(b) Rhombus if AB = BC = CD = DA and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if AB = DC, BC = AD; $AC \neq BD$; $AC \not\perp BD$

(d) Rectangle if AB = CD, BC = DA, AC = BD ; $AC \not\perp BD$

Illustration 1 : The number of points on x-axis which are at a distance $c (c < 3)$ from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

Solution : Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$$= \sqrt{(x_1-2)^2+9} = c \quad \text{or} \quad (x_1-2)^2 = c^2-9$$

$$\therefore\ x_1 - 2 = \pm\sqrt{c^2-9} \ \text{ since } c < 3 \Rightarrow c^2 - 9 < 0$$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Illustration 2 : The distance between the point $P(a\cos\alpha,\ a\sin\alpha)$ and $Q(a\cos\beta,\ a\sin\beta)$, where $a > 0$ & $\alpha > \beta$, is -

(A) $4a\sin\frac{\alpha-\beta}{2}$ (B) $2a\sin\frac{\alpha+\beta}{2}$ (C) $2a\sin\frac{\alpha-\beta}{2}$ (D) $2a\cos\frac{\alpha-\beta}{2}$

Solution :

$$d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha-\cos\beta)^2 + a^2(\sin\alpha-\sin\beta)^2$$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = 2a\sin\frac{\alpha-\beta}{2}$$

Ans. (C)

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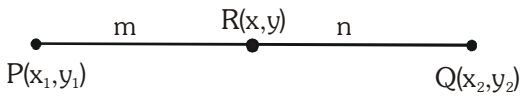
5. SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

(a) For internal division : P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x, y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



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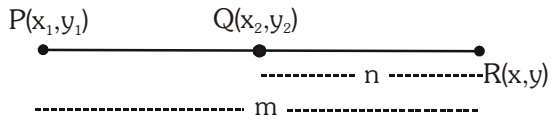
(b) **For external division :** R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$



$\frac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\frac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

(c) **Harmonic conjugate :** If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB. Mathematically ; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in **H.P.**

Illustration 3 : Determine the ratio in which $y - x + 2 = 0$ divides the line joining (3, –1) and (8, 9).

Solution : Suppose the line $y - x + 2 = 0$ divides the line segment joining A(3, –1) and B(8, 9) in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$\Rightarrow \quad 9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow \quad 3\lambda - 2 = 0$ or $\lambda = \frac{2}{3}$

So, the required ratio is $\frac{2}{3} : 1$, i.e., 2 : 3 (internally) since here λ is positive.

Do yourself - 1 :

1. Find the distance between the points P(–3, 2) and Q(2, –1).

2. If the distance between the points P(–3, 5) and Q(–x, – 2) is $\sqrt{58}$, then find the value(s) of x.

3. A line segment is of the length 15 units and one end is at the point (3, 2), if the abscissa of the other end is 15, then find possible ordinates.

4. A triangle with vertices (4, 0), (–1, –1), (3, 5) is
(A) isosceles and right angled (B) isosceles but not right angled
(C) right angled but not isosceles (D) neither right angled nor isosceles

5. Find the value of x_1 if the distance between the points $(x_1, 2)$ and (3, 4) be 8.
(A) $3 \pm 2\sqrt{15}$ (B) $3 \pm \sqrt{15}$ (C) $2 \pm 3\sqrt{15}$ (D) $2 \pm \sqrt{15}$

6. The four points whose co-ordinates are (2,1),(1,4),(4,5),(5,2) form :
(A) a rectangle which is not a square (B) a trapezium which is not a parallelogram
(C) a square (D) a rhombus which is not a square

7. Find the co-ordinates of the point dividing the join of A(1, – 2) and B(4, 7) :

(a) Internally in the ratio 1 : 2 (b) Externally in the ratio of 2 : 1

8. In what ratio is the line joining A(8, 9) and B(– 7, 4) divided by

(a) the point (2, 7) (b) the x-axis (c) the y-axis.

9. If A and B are the points (–3,4) and (2,1), then the co-ordinates of the point C on AB produced such that AC = 2BC are :

(A) (2,4) (B) (3,7) (C) (7,–2) (D) $\left(-\frac{1}{2},\frac{5}{2}\right)$

10. Determine the ratio in which the point P(3 , 5) divides the join of A(1, 3) & B(7, 9). Find the harmonic conjugate of P w.r.t. A & B.

11. If the line 2x + y = k passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals :

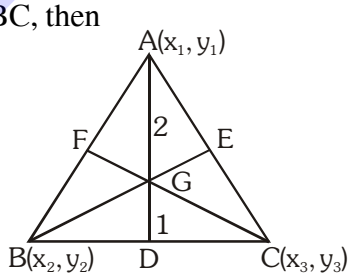
(A) $\frac{11}{5}$ (B) $\frac{29}{5}$ (C) 5 (D) 6

6. CO-ORDINATES OF SOME PARTICULAR POINTS :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

(a) Centroid :

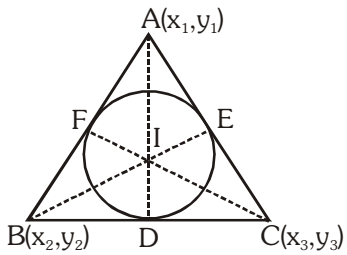
The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.



Co-ordinates of centroid $G\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)$

(b) Incenter :

The incenter is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.



Co-ordinates of incenter $I\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.

Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g.

$$\frac{BD}{DC}=\frac{AB}{AC}=\frac{c}{b}$$

(ii) Incenter divides the angle bisectors in the ratio $(b+c):a$, $(c+a):b$, $(a+b):c$.

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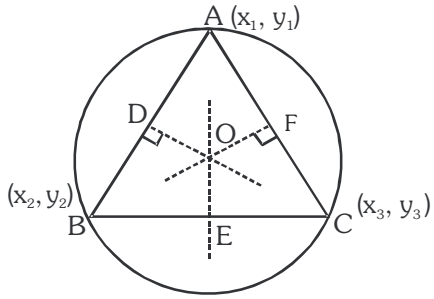
(c) Circumcenter :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcenter of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.



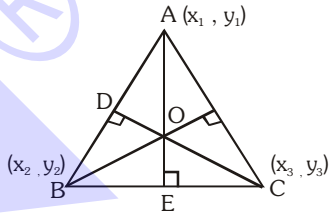
Note :

(i) If the triangle is right angled, then its circumcenter is the mid point of hypotenuse.

(ii) Co-ordinates of circumcenter $\left(\dfrac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \dfrac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$

(d) Orthocenter :

It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.



Note :

(i) If a triangle is right angled, then orthocenter is the point where right angle is formed.

(ii) Co-ordinates of orthocenter $\left(\dfrac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \dfrac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

Remarks :

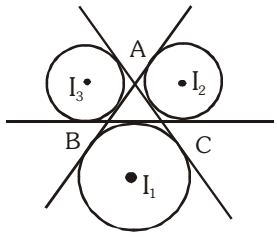
(i) If the triangle is equilateral, then centroid, incentre, orthocenter, circumcenter, coincide.

(ii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1

(iii) In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.

(e) Ex-centers :

The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates



are $I_1\left(\dfrac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \dfrac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$I_2\left(\dfrac{ax_1 - bx_2 + cx_3}{a-b+c}, \dfrac{ay_1 - by_2 + cy_3}{a-b+c}\right), I_3\left(\dfrac{ax_1 + bx_2 - cx_3}{a+b-c}, \dfrac{ay_1 + by_2 - cy_3}{a+b-c}\right)$

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Illustration 4 : If $\left(\frac{5}{3}, 3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Solution : Let the third vertex of triangle be (x, y), then

$\frac{5}{3} = \frac{x+0+2}{3} \Rightarrow x = 3$ and $3 = \frac{y+1+3}{3} \Rightarrow y = 5$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then AP = BP = CP = R (circumradius) $\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2 + (k-1)^2 = (h-2)^2 + (k-3)^2 = (h-3)^2 + (k-5)^2 = R^2$ ..... (i)

from the first two equations, we have

$h + k = 3$ ..... (ii)

from the first and third equation, we obtain

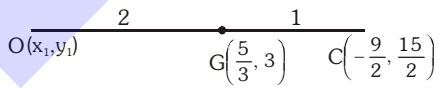
$6h + 8k = 33$ ..... (iii)

On solving, (ii) & (iii), we get

$h = -\frac{9}{2}, \; k = \frac{15}{2}$

Substituting these values in (i), we have

$R = \frac{5}{2}\sqrt{10}$



Let $O(x_1, y_1)$ be the orthocentre, then $\frac{x_1 + 2\left(-\frac{9}{2}\right)}{3} = \frac{5}{3} \Rightarrow x_1 = 14, \frac{y_1 + 2\left(\frac{15}{2}\right)}{3} = 3$

$\Rightarrow$ $y_1 = -6$. Hence orthocentre of the triangle is (14, –6).

Illustration 5 : The vertices of a triangle are A(0, –6), B(–6, 0) and C(1,1) respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\frac{-3}{2}, \frac{-3}{2}\right)$ (B) $\left(-4, \frac{3}{2}\right)$ (C) $\left(\frac{-3}{2}, \frac{3}{2}\right)$ (D) (–4, 6)

Solution : $a = BC = \sqrt{(-6-1)^2 + (0-1)^2} = \sqrt{50} = 5\sqrt{2}$

$b = CA = \sqrt{(1-0)^2 + (1+6)^2} = \sqrt{50} = 5\sqrt{2}$

$c = AB = \sqrt{(0+6)^2 + (-6-0)^2} = \sqrt{72} = 6\sqrt{2}$

coordinates of ex-centre opposite to vertex A will be :

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$$x = \frac{-ax_1 + bx_2 + cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0 + 5\sqrt{2}(-6) + 6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1 + by_2 + cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6) + 5\sqrt{2}.0 + 6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence coordinates of ex-centre is (–4, 6) **Ans. (D)**

Do yourself - 2 :

1. The coordinates of the vertices of a triangle are (1, 2), (3, 4) and (4, 6) :
(a) Find centroid of the triangle.
(b) Find circumcentre & the circumradius.
(c) Find orthocentre of the triangle.

2. A triangle has a vertex at (1, 2) and the mid points of the two sides through it are (–1, 1) and (2,3). Then the centroid of this triangle is :
(A) $\left(\frac{1}{3},1\right)$ (B) $\left(\frac{1}{3},2\right)$ (C) $\left(1,\frac{7}{3}\right)$ (D) $\left(\frac{1}{3},\frac{5}{3}\right)$

3. If a vertex of a triangle is (1, 1) and the mid points of two sides through this vertex are (–1, 2) and (3, 2) then the centroid of the triangle is
(A) $\left(-1,\frac{7}{3}\right)$ (B) $\left(\frac{-1}{3},\frac{7}{3}\right)$ (C) $\left(1,\frac{7}{3}\right)$ (D) $\left(\frac{1}{3},\frac{7}{3}\right)$

4. Coordinates of the vertices of a triangle ABC are (12,8), (–2,6) and (6,0) then the **correct** statement is -
(A) triangle is right but not isosceles
(B) triangle is isosceles but not right
(C) triangle is obtuse
(D) the product of the abscissa of the centroid, orthocenter and circumcenter is 160.

5. If the two vertices of a triangle are (7,2) and (1,6) and its centroid is (4,6) then the coordinate of the third vertex are (a,b). The value of (a + b), is-
(A) 13 (B) 14 (C) 15 (D) 16

6. The orthocenter of the triangle ABC is 'B' and the circumcenter is 'S' (a,b). If A is the origin then the co-ordinates of C are :
(A) (2a,2b) (B) $\left(\frac{a}{2},\frac{b}{2}\right)$ (C) $\left(\sqrt{a^2+b^2},0\right)$ (D) none

7. The medians of a triangle meet at (0,–3) and its two vertices are at (–1,4) and (5,2). Then the third vertex is at -
(A) (4,15) (B) (–4,–15) (C) (–4,15) (D) (4,–15)

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8. Line $\frac{x}{6}+\frac{y}{8}=1$ intersects the x and y axes at M and N respectively. If the coordinates of the point P lying inside the triangle OMN (where 'O' is origin) are (a, b) such that the areas of the triangle POM, PON and PMN are equal. Find

(a) the coordinates of the point P and

(b) the radius of the circle escribed opposite to the angle N.

9. Let A (-3, 2) and B (-2,1) be the vertices of a triangle ABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then the vertex C lies on the line

(1) $4x + 3y + 5 = 0$ (2) $3x + 4y + 5 = 0$

(3) $3x + 4y + 3 = 0$ (4) $4x + 3y + 3 = 0$

7. AREA OF TRIANGLE :

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]\right|$$

To remember the above formula, take the help of the following method :

$$=\frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]\right|$$

Remarks :

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) **Area of Equilateral triangle :** If altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$. If 'a' be the side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$.

(iii) Area of quadrilateral with given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$

$$\text{Area of quad. ABCD} = \frac{1}{2}\left|\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{bmatrix}\right|$$

Note : Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

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Illustration 6 : If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then its area is

(A) $\frac{25}{2}$ sq. units (B) 12 sq. units (C) 5 sq. units (D) 25 sq. units

Solution : $\Delta = \frac{1}{2}\left[1(-6-5)+4(5-2)+3(2+6)\right] = \frac{1}{2}\left[-11+12+24\right] = \frac{25}{2}$ square units

Ans. (A)

Illustration 7 : The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k:1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals -

(A) 7, 9 (B) 6, 7 (C) 7, $\frac{31}{9}$ (D) 9, $\frac{31}{9}$

Solution : $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of $\Delta ABC = 2$ units $\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right] = \pm 2$

$\Rightarrow 14k-66 = \pm 4(k+1) \Rightarrow k = 7$ or $\frac{31}{9}$

Ans. (C)

Illustration 8 : Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Solution : Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let x_1, y_1, x_2, y_2, x_3, y_3 be all rational.

Now area of $\Delta ABC = \frac{1}{2}|x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)| =$ Rational ..... (i)

Since ΔABC is equilateral

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2 = \frac{\sqrt{3}}{4}\{(x_1-x_2)^2+(y_1-y_2)^2\} =$ Irrational ..... (ii)

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence x_1, y_1, x_2, y_2, x_3, y_3 cannot all be rational.

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8. CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS :

Three given points A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

Do yourself - 3 :

1. Find the area of the triangle whose vertices are A(1,1), B(7, – 3) and C(12, 2)

2. Find the area of the quadrilateral whose vertices are A(1,1) B(7, – 3), C(12,2) and D(7, 21)

3. Prove that the points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear (By determinant method)

4. Let A(h, k), B(1, 1) and C(2, 1) be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1 square unit, then the set of values which 'k' can take is given by

(A) {–1, 3} (B) {–3, –2} (C) {1, 3} (D) {0, 2}

5. In a triangle ABC, coodinates of A are (1, 2) and the equations of the medians through B and C are x + y = 5 and x = 4 respectively. Then area of ΔABC (in sq. units) is

(A) 5 (B) 9 (C) 12 (D) 4

6. The point A divides the join of P(–5, 1) & Q(3, 5) in the ratio K : 1. Find the two values of K for which the area of triangle ABC, where B is (1, 5) & C is (7, –2), is equal to 2 units in magnitude.

7. The area of a triangle is 5. Two of its vertices are (2, 1) & (3, –2). The third vertex lies on y = x + 3. Find the third vertex.

9. LOCUS :

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) Equation of Locus :

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

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(b) Procedure for finding the equation of the locus of a point :

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminate contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminate. The resulting equation would be the equation of the locus of P.

Illustration 9 : The ends of the rod of length ℓ moves on two mutually perpendicular lines, find the locus of the point on the rod which divides it in the ratio $m_1 : m_2$. Considering perpendicular lines a coordinate axis, m_1 from x-axis and m_2 from y-axis.

(A) $m_1^2x^2 + m_2^2y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$ (B) $(m_2x)^2 + (m_1y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

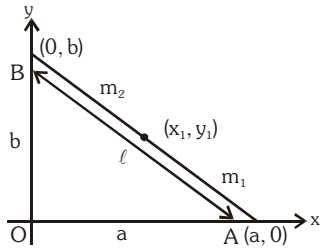
(C) $(m_1x)^2 + (m_2y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ (D) none of these

Solution : Let (h,k) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and OA = a, OB = b say

$\therefore \quad a^2 + b^2 = \ell^2$ ..... (i)

Now $h = \left(\dfrac{m_2a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)h$

$k = \left(\dfrac{m_1b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)k$



putting these values in (i) $\dfrac{(m_1+m_2)^2}{m_2^2}h^2 + \dfrac{(m_1+m_2)^2}{m_1^2}k^2 = \ell^2$

$\therefore$ Locus of (h,k) is $m_1^2x^2 + m_2^2y^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ **Ans. (C)**

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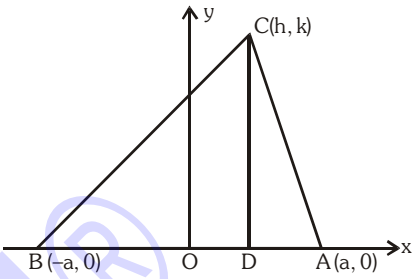
Illustration 10 : A(a, 0) and B(–a, 0) are two fixed points of ΔABC. If its vertex C moves in such a way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is -

(A) $y\lambda = 2a$ (B) $y = \lambda a$ (C) $ya = 2\lambda$ (D) none of these

Solution : Given that coordinates of two fixed points A and B are (a, 0) and (–a, 0) respectively. Let variable point C is (h, k). From the adjoining figure



$$\cot A = \frac{DA}{CD} = \frac{a-h}{k}$$

$$\cot B = \frac{BD}{CD} = \frac{a+h}{k}$$

But cotA + cotB = λ, so we have

$$\frac{a-h}{k} + \frac{a+h}{k} = \lambda \Rightarrow \frac{2a}{k} = \lambda$$

Hence locus of C is $y\lambda = 2a$ **Ans. (A)**

Do yourself - 4 :

1. Find the locus of a variable point which is at a distance of 2 units from the y-axis.

2. Find the locus of a point which is equidistant from both the axes.

3. A point P moves on the line 2x – 3y + 4 = 0. If Q(1,4) and R(3,–2) are fixed points, then the locus of the centroid of ΔPQR is a line :

(A) parallel to x-axis (B) with slope $\frac{2}{3}$

(C) with slope $\frac{3}{2}$ (D) parallel to y-axis

4. Let O(0, 0) and A(0, 1) be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4, is :

(A) $8x^2 - 9y^2 + 9y = 18$ (B) $9x^2 + 8y^2 - 8y = 16$

(C) $8x^2 + 9y^2 - 9y = 18$ (D) $9x^2 - 8y^2 + 8y = 16$

5. If the equation of the locus of a point equidistant from the point (a_1, b_1) and (a_2, b_2) is $(a_1 - b_2)x + (a_1 - b_2)y + c = 0$, then the value of 'c' is

(A) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$ (B) $\frac{1}{2}a_2^2 + b_2^2 - a_1^2 - b_1^2$

(C) $a_1^2 - a_2^2 + b_1^2 - b_2^2$ (D) $\frac{1}{2}\left(a_1^2 + a_2^2 + b_1^2 + b_2^2\right)$

6. Locus of centroid of the triangle whose vertices are (acost, asint), (bsint, –bcost) and (1, 0), where t is a parameter, is

(A) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$ (B) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$

(C) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$ (D) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

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7. Find the equation to the locus of a point which moves so that its distance from the axis of x is three times its distance from the axis of y.

8. Find the equation to the locus of a point which moves so that the sum of the squares of its distances from the axes is equal to 3.

9. Find the equation to the locus of a point which moves so that its distance from the point (3, 0) is three times its distance from (0, 2).

10. Find the equation to the locus of a point which moves so that its distance from the axis of x is always one half its distance from the origin.

10. STRAIGHT LINE :

Introduction : A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. **ax + by + c = 0 ; a & b ≠ 0** simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is **y = a** or **y = – a.**

(b) Equation of x-axis is **y = 0.**

(c) Equation of a line parallel to y-axis at a distance 'b' is **x = b** or **x = – b.**

(d) Equation of y-axis is **x = 0.**

Illustration 11 : Prove that every first degree equation in x, y represents a straight line.

Solution : Let ax + by + c = 0 be a first degree equation in x, y

where a, b, c are constants.

Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by ax + by + c = 0.

Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$

Let R be any point on the line segment joining P & Q

Suppose R divides PQ in the ratio λ : 1. Then, the coordinates of R are

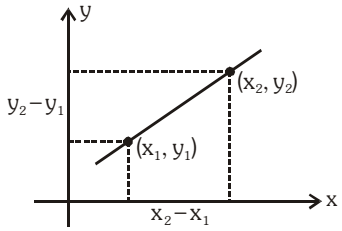
$$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \lambda\, 0 + 0 = 0$

$\therefore R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by ax + by + c = 0. Thus every point on the line segment joining P & Q lies on ax + by + c = 0.

Hence ax + by + c = 0 represents a straight line.

11. SLOPE OF LINE :



If a given line makes an angle **$\theta(0° \le \theta < 180°, \theta \ne 90°)$** with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter **m** i.e. **$m = \tan\theta$**. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \ne x_2$ then slope of line AB = $\dfrac{y_2 - y_1}{x_2 - x_1}$

Remark :

(i) If $\theta = 90°$, **m does not exist** and line is parallel to **y-axis.**

(ii) If $\theta = 0°$, **$m = 0$** and the line is parallel to **x-axis.**

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)

(a) If lines are parallel, **$m_1 = m_2$** and vice-versa.

(b) If lines are perpendicular, **$m_1 m_2 = -1$** and vice-versa

12. STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE :

(a) Slope Intercept form : Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $y = mx + c$

If the line passes through origin, its equation is written as $y = mx$

(b) Point Slope form : If m be the slope of a line and it passes through a point (x_1, y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

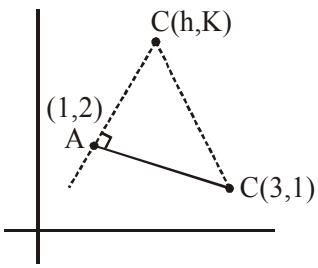
(c) Two point form : Equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Illustration 12 : A triangle ABC lying in the first quadrant has two vertices as A(1, 2) and B(3, 1). If $\angle BAC = 90°$, and ar(ΔABC) = $5\sqrt{5}$ sq. units, then the abscissa of the vertex C is :

(A) $2 + \sqrt{5}$ (B) $1 + \sqrt{5}$ (C) $1 + 2\sqrt{5}$ (D) $2\sqrt{5} - 1$

Solution : $\left(\dfrac{K-2}{h-1}\right)\left(\dfrac{1-2}{3-1}\right) = -1 \Rightarrow K = 2h$ ...(1)



$\sqrt{5}\,|h - 1| = 10$

$\because$ [ΔABC] = $5\sqrt{5}$

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$\Rightarrow \frac{1}{2}\left(\sqrt{5}\right)\sqrt{(h-1)^2+(K-2)^2} = 5\sqrt{5}$ ....(2)

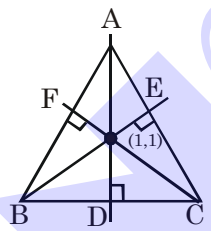
$\Rightarrow$ h = $2\sqrt{5}+1$ (h > 0) **Ans. (C)**

Illustration 13 : Let the equations of two sides of a triangle be 3x – 2y + 6 = 0 and 4x + 5y – 20 = 0. If the orthocentre of this triangle is at (1,1), then the equation of its third side is :

(A) 122y – 26x – 1675 = 0 (B) 26x + 61y + 1675 = 0

(C) 122y + 26x + 1675 = 0 (D) 26x – 122y – 1675 = 0

Solution : Equation of AB is



3x – 2y + 6 = 0

equation of AC is

4x + 5y – 20 = 0

Equation of BE is

2x + 3y – 5 = 0

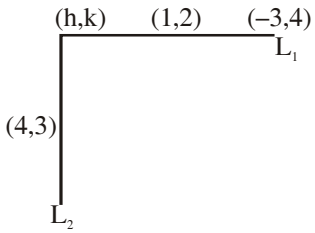
Equation of CF is 5x – 4y – 1 = 0

$\Rightarrow$ Equation of BC is 26x – 122y = 1675 **Ans. (D)**

Illustration 14 : Suppose that the points (h,k), (1,2) and (–3,4) lie on the line L_1. If a line L_2 passing through the points (h,k) and (4,3) is perpendicular to L_1, then $\frac{k}{h}$ equals :

(A) 3 (B) $-\frac{1}{7}$ (C) $\frac{1}{3}$ (D) 0

Solution : equation of L_1 is



$y = -\frac{1}{2}x + \frac{5}{2}$ ...(1)

equation of L_2 is

y = 2x – 5 ...(2)

by (1) and (2)

x = 3

y = 1 $\Rightarrow$ h = 3, k = 1

$\frac{k}{h} = \frac{1}{3}$ **Ans. (C)**

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Do yourself - 5 :

1. If the perpendicular bisector of the line segment joining the points P (1, 4) and Q (k, 3) has y-intercept equal to –4, then a value of k is :-

(A) $\sqrt{15}$ (B) –2 (C) $\sqrt{14}$ (D) –4

2. If in a parallelogram ABDC, the coordinates of A, B and C are respectively (1, 2), (3, 4) and (2, 5), then the equation of the diagonal AD is:-

(A) $5x + 3y - 11 = 0$ (B) $3x - 5y + 7 = 0$
(C) $3x + 5y - 13 = 0$ (D) $5x - 3y + 1 = 0$

3. If the straight line, $2x-3y+17 = 0$ is perpendicular to the line passing through the points (7, 17) and (15, β), then β equals :-

(A) –5 (B) $-\frac{35}{3}$ (C) $\frac{35}{3}$ (D) 5

4. If the two lines $x + (a - 1) y = 1$ and $2x + a^2y = 1 (a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is :-

(A) $\frac{2}{5}$ (B) $\frac{2}{\sqrt{5}}$ (C) $\frac{\sqrt{2}}{5}$ (D) $\sqrt{\frac{2}{5}}$

5. A point on the straight line, $3x + 5y = 15$ which is equidistant from the coordinate axes will lie only in :

(A) 1st and 2nd quadrants (B) 4th quadrant
(C) 1st, 2nd and 4th quadrant (D) 1st quadrant

6. The circumcentre of a triangle lies at the origin and its centroid is the mid point of the line segment joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$, $a \neq 0$. Then for any a, the orthocentre of this triangle lies on the line :

(A) $y - (a^2 + 1) x = 0$ (B) $y + x = 0$
(C) $(a - 1)^2x - (a + 1)^2 y = 0$ (D) $y - 2ax = 0$

7. If a line intercepted between the coordinate axes is trisected at a point A (4, 3), which is nearer to x-axis, then its equation is :-

(A) $3x + 8y = 36$ (B) $4x - 3y = 7$
(C) $x + 3y = 13$ (D) $3x + 2y = 18$

8. Given three points P, Q, R with P(5, 3) and R lies on the x-axis. If equation of RQ is $x - 2y = 2$ and PQ is parallel to the x-axis, then the centroid of ΔPQR lies on the line:

(A) $x - 2y + 1 = 0$ (B) $5x - 2y = 0$
(C) $2x + y - 9 = 0$ (D) $2x - 5y = 0$

9. If the straight lines $x + 3y = 4$, $3x + y = 4$ and $x + y = 0$ form a triangle, then the triangle is:-

(A) Scalene (B) Equilateral
(C) Isosceles (D) Right angled isosceles

10. If A(2, –3) and B(–2, 1) are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is :

(A) $2x - 3y = 1$ (B) $x - y = 1$ (C) $2x + 3y = 1$ (D) $2x + 3y = 3$

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11. Let A(2, –3) and B(–2, 3) be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line

(A) $3x - 2y = 3$ (B) $2x - 3y = 7$ (C) $3x + 2y = 5$ (D) $2x + 3y = 9$

12. If two vertices of a triangle are (5, –1) and (–2, 3) and its orthocentre is at (0, 0), then the third vertex is

(A) (4,–7) (B) (–4, –7) (C) (–4, 7) (D) (4, 7)

13. If two vertical poles 20 m and 80 m high stand apart on a horizontal plane, then the height (in m) of the point of intersection of the lines joining the top of each pole to the foot of other is :-

(A) 16 (B) 15 (C) 50 (D) 18

14. The perpendicular bisector of the line segment joining P (1, 4) and Q (k, 3) has y-intercept –4. Then a possible value of k is

(A) 1 (B) 2 (C) –2 (D) –4

15. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(A) are vertices of a triangle (B) lie on a straight line

(C) lie of an ellipse (D) lie on a circle

16. The diagonals of a parallelogram PQRS are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then PQRS must be a :

(A) rectangle (B) square

(C) cyclic quadrilateral (D) rhombus

17. Number of lines that can be drawn through the point(4,–5) so that its distance from (–2,3) will be equal to 12 is equal to-

(A) 0 (B) 1 (C) 2 (D) 3

18. A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain the equation.

19. The points (1, 3) & (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c & the remaining vertices.

20. Consider a triangle ABC with sides AB and AC having the equations $L_1 = 0$ and $L_2 = 0$. Let the centroid, orthocentre and circumcentre of the ΔABC are G, H and S respectively. $L = 0$ denotes the equation of sides BC.

(a) If $L_1 : 2x - y = 0$ and $L_2 : x + y = 3$ and G(2, 3) then find the slope of the line $L = 0$.

(b) If $L_1 : 2x + y = 0$ and $L_2 : x - y + 2 = 0$ and H(2, 3) then find the y-intercept of $L = 0$.

(c) If $L_1 : x + y - 1 = 0$ and $L_2 : 2x - y + 4 = 0$ and S(2, 1) then find the x-intercept of the line $L = 0$.

21. Two equal sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ & its third side passes through the point (1, –10). Determine the equation of the third side.

22. Two consecutive sides of a parallelogram are $4x + 5y = 0$ & $7x + 2y = 0$. If the equation to one diagonal is $11x + 7y = 9$, find the equation to the other diagonal.

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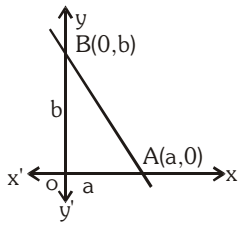
(d) **Intercept form :** If a and b are the intercepts made by a line on the axes of x and y, its equation is written as : $\frac{x}{a}+\frac{y}{b}=1$



(i) Length of intercept of line between the coordinate axes $=\sqrt{a^2+b^2}$

(ii) Area of triangle AOB $=\frac{1}{2}OA.OB=\left|\frac{1}{2}ab\right|$

Illustration 15 : The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24$, $x - y = 7$ (B) $4x + 3y = 24$, $x + y = 7$

(C) $4x + 3y + 24 = 0$, $x + y + 7=0$ (D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution : Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ .....(i)

This passes through (3 , 4) , therefore $\frac{3}{a}+\frac{4}{b}=1$ .....(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii) , we get

$\frac{3}{a}+\frac{4}{14-a}=1 \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7 , 6$

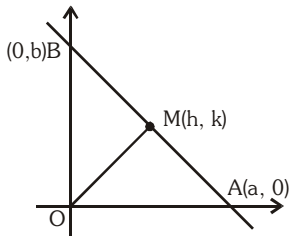
For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i) , we get the equations of the lines

$\frac{x}{7}+\frac{y}{7}=1$ and $\frac{x}{6}+\frac{y}{8}=1$ or $x+ y = 7$ and $4x + 3y = 24$ **Ans. (B)**

Illustration 16 : Two points A and B move on the positive direction of x-axis and y-axis respectively, such that OA + OB = K. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.

Solution : Let the equation of AB be $\frac{x}{a}+\frac{y}{b}=1$ ..... (i)



given, $a + b = K$ ..... (ii)

now, $m_{AB} \times m_{OM} = -1 \Rightarrow ah = bk$ ..... (iii)

from (ii) and (iii),

$a = \frac{kK}{h+k}$ and $b = \frac{hK}{h+k}$

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$\therefore$ from (i) $\dfrac{x(h+k)}{k.K}+\dfrac{y(h+k)}{h.K}=1$

as it passes through (h, k)

$\dfrac{h(h+k)}{k.K}+\dfrac{k(h+k)}{h.K}=1 \Rightarrow \quad (h+k)(h^2+k^2)=Khk$

$\therefore$ locus of (h, k) is $(x+y)(x^2+y^2)=Kxy$.

(e) **Normal form :** If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $\mathbf{x\cos\alpha + y\sin\alpha = p}$ (p is always positive) where $\mathbf{0 \le \alpha < 2\pi}$.

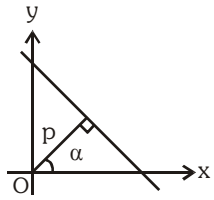
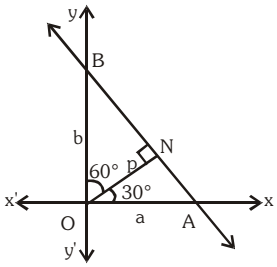


Illustration 17 : Find the equation of the straight line on which the perpendicular from origin makes an angle 30° with positive x-axis and which forms a triangle of area $\left(\dfrac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

Solution : $\angle NOA = 30°$

Let ON = p > 0, OA = a, OB = b



In ΔONA, $\cos 30° = \dfrac{ON}{OA}=\dfrac{p}{a} \Rightarrow \dfrac{\sqrt{3}}{2}=\dfrac{p}{a}$

or $\quad a = \dfrac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \dfrac{ON}{OB}=\dfrac{p}{b} \Rightarrow \dfrac{1}{2}=\dfrac{p}{b}$

or $\quad b = 2p$

$\because$ Area of $\Delta OAB = \dfrac{1}{2}\,ab = \dfrac{1}{2}\left(\dfrac{2p}{\sqrt{3}}\right)(2p)=\dfrac{2p^2}{\sqrt{3}}$

$\therefore \quad \dfrac{2p^2}{\sqrt{3}}=\dfrac{50}{\sqrt{3}} \Rightarrow \quad p^2 = 25$

or $\quad p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos 30° + y\sin 30° = 5$

or $\quad x\sqrt{3} + y = 10$

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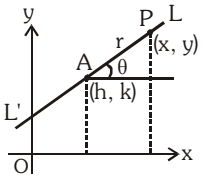
(f) **Parametric form :** To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r, then $\mathbf{x - h = r\cos\theta,\ y - k = r\sin\theta}$ & $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} =$ **r** is the equation of the straight line LAL'.

Any point P on the line will be of the form (h + r cosθ, k + r sinθ), where |r| gives the distance of the point P from the fixed point (h, k).

Illustration 18 : Equation of a line which passes through point A(2, 3) and makes an angle of 45° with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Solution : Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$ hence $\dfrac{x-2}{\cos 45°} = \dfrac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow$ $x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2+\dfrac{r}{\sqrt{2}}, 3+\dfrac{r}{\sqrt{2}}\right)$.

If this point is on line x + y + 1 = 0 then

$\left(2+\dfrac{r}{\sqrt{2}}\right)+\left(3+\dfrac{r}{\sqrt{2}}\right)+1=0 \quad \Rightarrow r = -3\sqrt{2}\ ;\ |r| = 3\sqrt{2}$ **Ans. (B)**

Illustration 19 : A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively. A point A is taken on the variable line such that $\dfrac{m+n}{OA} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution : Let the variable line passing through the origin is $\dfrac{x}{\cos\theta} = \dfrac{y}{\sin\theta} = r_i$ ..... (i)

Let the equation of the line L_1 is $p_1x + q_1y = 1$ ..... (ii)

Equation of the line L_2 is $p_2x + q_2y = 1$ ..... (iii)

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

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Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$$\Rightarrow \quad r_1 = OA_1 = \frac{1}{p_1\cos\theta + q_1\sin\theta}$$

Similarly, $r_2 = OA_2 = \dfrac{1}{p_2\cos\theta + q_2\sin\theta}$

Let OA = r

Let co-ordinate of A are (h, k) $\Rightarrow$ $(h, k) \equiv (r\cos\theta, r\sin\theta)$

Now $\dfrac{m+n}{r} = \dfrac{m}{OA_1} + \dfrac{n}{OA_2} \Rightarrow \dfrac{m+n}{r} = \dfrac{m}{r_1} + \dfrac{n}{r_2}$

$\Rightarrow \quad m + n = m(p_1r\cos\theta + q_1r\sin\theta) + n(p_2r\cos\theta + q_2r\sin\theta)$

$\Rightarrow \quad (p_1h + q_1k - 1) + \dfrac{n}{m}(p_2h + q_2k - 1) = 0$

Therefore, locus of A is $(p_1x+q_1y - 1) + \dfrac{n}{m}(p_2x + q_2y - 1) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$ where $\lambda = \dfrac{n}{m}$.

This is the equation of the line passing through the intersection of L_1 and L_2.

Illustration 20 : A straight line through P(–2, –3) cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if PQ. PR = 20.

Solution : Let line be $\dfrac{x+2}{\cos\theta} = \dfrac{y+3}{\sin\theta} = r$

$\Rightarrow \quad x = r\cos\theta - 2,\ y = r\sin\theta - 3$ ..... (i)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ ..... (ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need PQ . PR = $r_1 . r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta) + (\text{some terms})r + 80 = 0$

$$\therefore \quad r_1.r_2 = PQ.\ PR = \frac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$$

$\therefore \quad \cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ $\quad (\because$ PQ . PR = 20)

$\therefore \quad \sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow \quad (\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

$\therefore \quad \tan\theta = 1,\ \tan\theta = 3$

hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0$.

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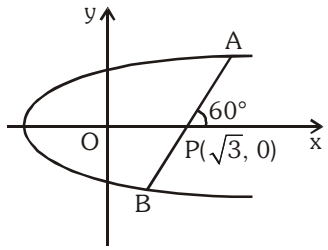
Illustration 21 : If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Solution : Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$.



If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore \quad \theta = 60°$

$\frac{x - \sqrt{3}}{\cos 60°} = \frac{y - 0}{\sin 60°} = r \Rightarrow \left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ be a point on the parabola $y^2 = x + 2$

then $\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2 \Rightarrow 3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$\therefore \quad PA.PB = r_1 r_2 = \left|\frac{-4(2+\sqrt{3})}{3}\right| = \frac{4(2+\sqrt{3})}{3}$

Do yourself - 6 :

1. Reduce the line $2x - 3y + 5 = 0$,

(a) In slope- intercept form and hence find slope & Y-intercept

(b) In intercept form and hence find intercepts on the axes.

(c) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

2. Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$.

3. If a straight line passing thourgh the point P(–3, 4) is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is :

(A) $x - y + 7 = 0$ (B) $3x - 4y + 25 = 0$

(C) $4x + 3y = 0$ (D) $4x - 3y + 24 = 0$

4. A square of each side 2, lies above the x-axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30º with the positive direction of the x-axis, then the sum of the x-coordinates of the vertices of the square is :

(A) $\sqrt{3} - 2$ (B) $\sqrt{3} - 1$ (C) $2\sqrt{3} - 1$ (D) $2\sqrt{3} - 2$

5. A straight line through the point A(3, 4) is such that its intercept between the axes is bisected at A. Its equation is

(A) $x + y = 7$ (B) $3x - 4y + 7 = 0$

(C) $4x + 3y = 24$ (D) $3x + 4y = 25$

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6. The equation of the straight lines passing through the point (4, 3) and making intercepts on the co-ordinates axes whose sum is –1 is

(A) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$ (B) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$

(C) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$ (D) $\frac{x}{2}+\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$

7. Locus of mid point of the portion between the axes of $x\cos\alpha + y\sin\alpha = p$ where p is constant is

(A) $x^2+y^2=\frac{4}{p^2}$ (B) $x^2+y^2=4p^2$ (C) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$ (D) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$

8. Area of the quadrilateral formed by the lines $|x|+|y|=2$ is :

(A) 8 (B) 6 (C) 4 (D) none

9. If the x intercept of the line $y = mx + 2$ is greater than 1/2 then the gradient of the line lies in the interval-

(A) (–1,0) (B) (–1/4,0) (C) (–∞,–4) (D) (–4,0)

10. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is :

(1) $-\frac{1}{2}$ (2) $-\frac{1}{4}$ (3) –4 (4) –2

11. If the x-intercept of some line L is double as that of the line, $3x + 4y = 12$ and the y-intercept of L is half as that of the same line, then the slope of L is :-

(1) –3 (2) –3/2 (3) –3/8 (4) –3/16

(g) **General form :** We know that a first degree equation in x and y, $ax + by + c = 0$ always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $=\frac{-a}{b}=-\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $=-\frac{c}{a}$ and intercept by this line on y-axis $=-\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{a^2+b^2}$.

13. EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE :

(a) Equation of line parallel to line $ax + by + c = 0$

$\mathbf{ax + by + \lambda = 0}$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$\mathbf{bx - ay + k = 0}$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

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14. ANGLE BETWEEN TWO LINES :

(a) If θ be the angle between two lines : $y = m_1x + c_1$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

Note :

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of tanθ.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the ΔABC found by these formulas are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2};\ \tan B = \frac{m_2 - m_3}{1 + m_2 m_3}\ \&\ \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these lines are -

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Illustration 22 : If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is -

(A) 3 (B) 4 (C) –1 (D) –4

Solution : $m_1 = -\frac{1}{4}$ $m_2 = -\frac{4}{k}$

Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right) \times \left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1$ **Ans. (C)**

Illustration 23 : A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is -

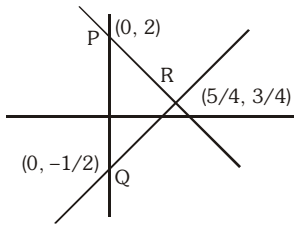
(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

Solution : Equation of the line L is $y - 1 = \frac{-1}{1}(x - 1) \Rightarrow y = -x + 2$

Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P and Q, then PQ = 5/2.

Also x-coordinate of their point of intersection R = 5/4



$\therefore$ area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2} \times \frac{5}{4}\right) = 25/16.$ **Ans. (B)**

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Illustration 24 : If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$

Ans. (D)

15. STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equation of line passing through a point (x_1, y_1) and making an angle α, with the line y=mx+c is written as :

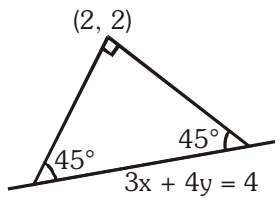
$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Illustration 25 : Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Solution : The problem can be restated as :

Find the equations of the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -3/4$.

$$\Rightarrow \quad \tan 45° = \pm\frac{m - m_1}{1 + m_1 m}, \text{ i.e., } 1 = \pm\frac{m + 3/4}{1 - \frac{3}{4}m}$$



$$m_A = \frac{1}{7}, \text{ and } m_B = -7$$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$ **Ans.**

Do yourself - 7 :

1. Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$.

2. Find the line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

3. Find the equation of the line which has positive y-intercept 4 units and is parallel to the line $2x - 3y - 7 = 0$. Also find the point where it cuts the x-axis.

4. Classify the following pairs of lines as coincident, parallel or intersecting :

(a) $x + 2y - 3 = 0$ & $-3x - 6y + 9 = 0$

(b) $x + 2y + 1 = 0$ & $2x + 4y + 3 = 0$

(c) $3x - 2y + 5 = 0$ & $2x + y - 5 = 0$

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5. Consider the straight lines

$L_1 : x - y = 1$

$L_2 : x + y = 1$

$L_3 : 2x + 2y = 5$

$L_4 : 2x - 2y = 7$

The correct statement is :

(A) $L_1 \perp L_2$, $L_2 \parallel L_3$, L_1 intersects L_4
(B) $L_1 \perp L_2$, $L_1 \parallel L_3$, L_1 intersects L_2
(C) $L_1 \perp L_2$, $L_1 \perp L_3$, L_2 intersects L_4
(D) $L_1 \parallel L_4$, $L_2 \parallel L_3$, L_2 intersects L_3

6. A straight line L through the point (3,–2) is inclined at an angle of 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x-axis, then the equation of L is :-

(A) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$
(B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(C) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$
(D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

7. If a, b, c $\in$ R and 1 is a root of the equation $ax^2 + bx + c = 0$, then the curve $y = 4ax^2 + 3bx + 2c$, $a \neq 0$ intersects x-axis at

(A) no point
(B) exactly two distinct point
(C) exactly one point
(D) two distinct points whose coordinates are always rational numbers

8. The point of intersection of the lines $(a^3 + 3)x + ay + a - 3 = 0$ and $(a^5 + 2)x + (a + 2)y + 2a + 3 = 0$ (a real) lies on the y-axis for :-

(A) No value of a
(B) Exactly two values of a
(C) More than two values of a
(D) Exactly one value of a

9. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for :

(A) exactly one values of p
(B) exactly two values of p
(C) more than two values of p
(D) no value of p

10. A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L & the coordinate axes is 5. Find the equation of the line.

16. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

Length of perpendicular from a point (x_1, y_1) on the line $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \dfrac{|c|}{\sqrt{a^2 + b^2}}$

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Illustration 26 : If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution : Let n given points be (x_i, y_i) where i = 1, 2.... n and the variable straight line is $ax + by + c = 0$.

Given that $\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2+b^2}}\right) = 0 \Rightarrow a\Sigma x_i + b\Sigma y_i + cn = 0 \Rightarrow a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\frac{\Sigma x_i}{n}, \frac{\Sigma y_i}{n}\right)$. **Ans.**

Illustration 27 : Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.

Solution : Suppose, if possible.

Equation of line through (4, –5) with slope m is $y + 5 = m(x - 4)$

$\Rightarrow \quad mx - y - 4m - 5 = 0$

Then $\frac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2+1}} = 12$

$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2+1)}$

On squaring, $\quad (6m + 8)^2 = 144(m^2 + 1)$

$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1) \Rightarrow (3m + 4)^2 = 36(m^2 + 1)$

$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \quad$ ..... (i)

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

17. DISTANCE BETWEEN TWO PARALLEL LINES :

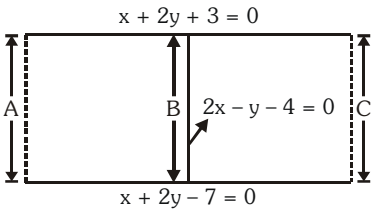
(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}}$

(**Note :** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Illustration 28 : Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution : Distance between the two parallel lines is $\frac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.



The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B = distance between sides B and C

$$\Rightarrow \frac{|k-(-4)|}{\sqrt{5}} = 2\sqrt{5} \Rightarrow \frac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \Rightarrow k = 6, -14.$$

Hence the fourth sides of the two squares are (i) $2x - y + 6 = 0$ (ii) $2x - y - 14 = 0$.

Ans.

Do yourself - 8 :

1. Find the distances between the following pair of parallel lines :

(a) $3x + 4y = 13$, $3x + 4y = 3$

(b) $3x - 4y + 9 = 0$, $6x - 8y - 15 = 0$

2. Find the points on the x-axis such that their perpendicular distance from the line $\frac{x}{a} + \frac{y}{b} = 1$ is 'a', $a, b > 0$.

3. Show that the area of the parallelogram formed by the lines

$2x - 3y + a = 0$, $3x - 2y - a = 0$, $2x - 3y + 3a = 0$ and $3x - 2y - 2a = 0$ is $\frac{2a^2}{5}$ square units.

4. If the line, $2x - y + 3 = 0$ is at a distance $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ from the lines $4x - 2y + \alpha = 0$ and $6x - 3y + \beta = 0$, respectively, then the sum of all possible values of α and β is _____

5. A straight line through origin O meets the lines $3y = 10 - 4x$ and $8x + 6y + 5 = 0$ at points A and B respectively. Then O divides the segment AB in the ratio :

(A) 2 : 3 (B) 1 : 2 (C) 4 : 1 (D) 3 : 4

6. The point (2, 1) is translated parallel to the line $L : x - y = 4$ by $2\sqrt{3}$ units. If the new point Q lies in the third quadrant, then the equation of the line passing through Q and perpendicular to L is :

(A) $x + y = 3 - 3\sqrt{6}$ (B) $x + y = 3 - 2\sqrt{6}$

(C) $x + y = 2 - \sqrt{6}$ (4) $2x + 2y = 1 - \sqrt{6}$

7. If a line L is perpendicular to the line $5x - y = 1$, and the area of the triangle formed by the line L and the coordinate axes is 5, then the distance of line L from the line $x + 5y = 0$ is:-

(1) $\frac{7}{\sqrt{5}}$ (2) $\frac{5}{\sqrt{13}}$ (3) $\frac{7}{\sqrt{13}}$ (4) $\frac{5}{\sqrt{7}}$

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8. The base of an equilateral triangle is along the line given by $3x + 4y = 9$. If a vertex of the triangle is (1, 2), then the length of a side of the triangle is :

(A) $\frac{4\sqrt{3}}{15}$ (B) $\frac{4\sqrt{3}}{5}$ (C) $\frac{2\sqrt{3}}{15}$ (D) $\frac{2\sqrt{3}}{5}$

9. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is :-

(A) $\frac{3\sqrt{2}}{5}$ (B) $\frac{\sqrt{3}}{4}$ (C) $\frac{3\sqrt{2}}{8}$ (D) $\frac{2\sqrt{3}}{8}$

18. POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE :

Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

19. CONCURRENCY OF LINES :

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent,

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Illustration 29 : If the lines $ax + by + p = 0$, $x\cos\alpha + y\sin\alpha - p = 0$ $(p \neq 0)$ and $x\sin\alpha - y\cos\alpha = 0$ are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then $a^2 + b^2$ is equal to -

(A) 1 (B) 2 (C) 4 (D) p^2

Solution : Since the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0$ ..... (i)

As $ax + by + p = 0$ and $x\cos\alpha + y\sin\alpha - p = 0$ include an angle $\frac{\pi}{4}$.

$$\pm\tan\frac{\pi}{4}=\frac{-\frac{a}{b}+\frac{\cos\alpha}{\sin\alpha}}{1+\frac{a}{b}\frac{\cos\alpha}{\sin\alpha}}$$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm(b\sin\alpha + a\cos\alpha)$

$\Rightarrow$ $-a\sin\alpha + b\cos\alpha = \pm 1$ [from (i)] ..... (ii)

Squaring and adding (i) & (ii), we get

$a^2 + b^2 = 2.$ **Ans. (B)**

Do yourself - 9 :

1. Examine the positions of the points (3, 4) and (2, – 6) w.r.t. $3x - 4y = 8$

2. If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.

3. Find the equation of the line joining the point (2, – 9) and the point of intersection of lines $2x + 5y - 8 = 0$ and $3x - 4y - 35 = 0$.

4. Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

5. The set of all possible values of θ in the interval $(0, \pi)$ for which the points (1, 2) and $(\sin\theta, \cos\theta)$ lie on the same side of the line $x + y = 1$ is :

(A) $\left(0,\frac{\pi}{4}\right)$ (B) $\left(0,\frac{3\pi}{4}\right)$ (C) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ (D) $\left(0,\frac{\pi}{2}\right)$

6. If the point (1, a) lies in between the straight lines $x + y = 1$ and $2(x + y) = 3$ then a lies in interval :-

(A) $\left(1,\frac{3}{2}\right)$ (B) $\left(0,\frac{1}{2}\right)$ (C) $(-\infty, 0)$ (D) $\left(\frac{3}{2},\infty\right)$

7. If (a, a^2) falls inside the angle made by the lines $y=\frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belong to

(A) $\left(0,\frac{1}{2}\right)$ (B) $(3, \infty)$ (C) $\left(\frac{1}{2},3\right)$ (D) $\left(-3,-\frac{1}{2}\right)$

8. If the three distinct lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4ay + a = 0$ are concurrent, then the point (a, b) lies on a :-

(1) circle (2) straight line (3) parabola (4) hyperbola

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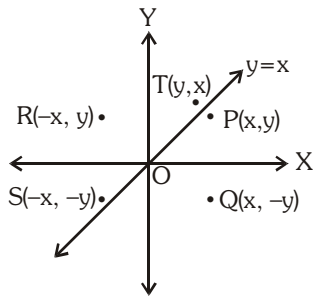
20. REFLECTION OF A POINT :



Let P(x, y) be any point, then its image with respect to

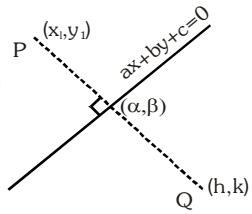
(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line ax + by + c = 0 is given by following formula.



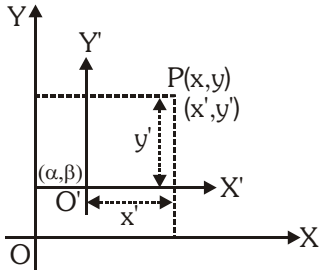
$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=-2\frac{(ax_1+by_1+c)}{a^2+b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

21. TRANSFORMATION OF AXES

(a) Shifting of origin without rotation of axes :



Let P (x, y) with respect to axes OX and OY.

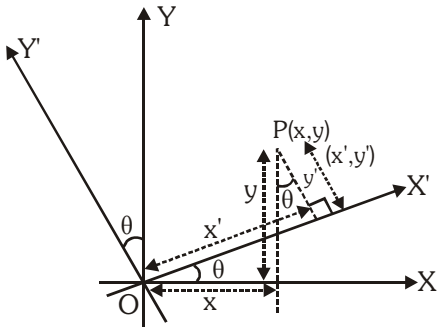
Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y' , where OX and O'X' are parallel and OY and O'Y' are parallel.

Then **x = x' + α, y = y' + β**

or **x' = x – α, y' = y – β**

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting x + α in place of x and y + β in place of y.

(b) Rotation of axes without shifting the origin :



Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where ∠X'OX = ∠YOY' = θ, where θ is measured in anticlockwise direction.

then $x = x' \cos\theta - y' \sin\theta$

$y = x' \sin\theta + y' \cos\theta$

and $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

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The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

New \ Old	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos\theta$	$\sin\theta$
$y' \rightarrow$	$-\sin\theta$	$\cos\theta$

Illustration 30 : Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution : Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}\left(x\cos\theta - y\sin\theta\right)\left(x\sin\theta + y\cos\theta\right) + 7(x\sin\theta + y\cos\theta)^2 = 10$$

$$\Rightarrow \quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta)$$
$$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of $xy = 0$)

We get $\quad -9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta = 0$

or $\quad \sin2\theta = -\sqrt{3}\cos2\theta \quad$ or $\quad \tan2\theta = -\sqrt{3} = \tan(180° - 60°)$

or $\quad 2\theta = 120° \quad \therefore \quad \theta = 60°$

Do yourself - 10 :

1. The point (4, 1) undergoes the following transformations, then the match the correct alternatives:

	Column-I		**Column-II**
(A)	Reflection about x-axis is	(p)	(4, –1)
(B)	Reflection about y-axis is	(q)	(–4, –1)
(C)	Reflection about origin is	(r)	$\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D)	Reflection about the line y = x is	(s)	(–4, 1)
(E)	Reflection about the line 4x + 3y – 5 = 0 is	(t)	(1, 4)

2. To what point must the origin be shifted, so that the coordinates of the point (4, 5) become (– 3, 9).

3. If the axes be turned through an angle $\tan^{-1}2$ (in anticlockwise direction),what does the equation $4xy - 3x^2 = a^2$ become ?

4. Let L denote the line in the xy-plane with x and y intercepts as 3 and 1 respectively. Then the image of the point (–1, –4) in this line is :

(A) $\left(\frac{8}{5}, \frac{29}{5}\right)$ (B) $\left(\frac{29}{5}, \frac{11}{5}\right)$ (C) $\left(\frac{11}{5}, \frac{28}{5}\right)$ (D) $\left(\frac{29}{5}, \frac{8}{5}\right)$

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5. A ray of light is incident along a line which meets another line, $7x - y + 1 = 0$, at the point (0, 1). The ray is then reflected from this point along the line, $y + 2x = 1$. Then the equation of the line of incidence of the ray of light is :-

(A) $41x - 25y + 25 = 0$ (B) $41x + 25y - 25 = 0$
(C) $41x + 38y - 38 = 0$ (D) $41x - 38y + 38 = 0$

6. Let L be the line passing through the point P(1, 2) such that its intercepted segment between the co-ordinate axes is bisected at P. If L_1 is the line perpendicular to L and passing through the point (–2, 1), then the point of intersection of L and L_1 is :

(A) $\left(\frac{3}{10}, \frac{17}{5}\right)$ (B) $\left(\frac{11}{20}, \frac{29}{10}\right)$ (C) $\left(\frac{4}{5}, \frac{12}{5}\right)$ (D) $\left(\frac{3}{5}, \frac{23}{10}\right)$

7. A ray of light coming from the point $\left(2, 2\sqrt{3}\right)$ is incident at an angle 30° on the line x=l at the point A. The ray gets reflected on the line x = 1 and meets x-axis at the point B. Then, the line AB passes through the point:

(A) $\left(3, -\frac{1}{\sqrt{3}}\right)$ (B) $\left(3, -\sqrt{3}\right)$ (C) $\left(4, -\frac{\sqrt{3}}{2}\right)$ (D) $\left(4, -\sqrt{3}\right)$

8. The foot of the perpendicular drawn from the origin, on the line, $3x + y = \lambda (\lambda \neq 0)$ is P. If the line meets x-axis at A and y–axis at B, then the ratio BP : PA is :-

(A) 9 : 1 (B) 1 : 3 (C) 3 : 1 (D) 1 : 9

9. A ray of light passing through the point A(1,2) is reflected at a point B on the x-axis and then passes through (5,3). Then the equation of AB is :

(A) $5x + 4y = 13$ (B) $5x - 4y = -3$ (C) $4x + 5y = 14$ (D) $4x - 5y = -6$

10. The equations of the perpendicular bisectors of the sides AB & AC of a triangle ABC are $x - y + 5 = 0$ & $x + 2y = 0$, respectively. If the point A is (1, –2) find the equation of the line BC.

11. Let P be the point (3, 2). Let Q be the reflection of P about the x-axis. Let R be the reflection of Q about the line $y = -x$ and let S be the reflection of R through the origin. PQRS is a convex quadrilateral. Find the area of PQRS.

12. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is :

(A) $y = x + \sqrt{3}$ (B) $\sqrt{3}y = x - \sqrt{3}$ (C) $y = \sqrt{3}x - \sqrt{3}$ (D) $\sqrt{3}y = x - 1$

13. If the image of point P(2, 3) in a line L is Q (4, 5) then, the image of point R (0, 0) in the same line is :

(A) (4, 5) (B) (2, 2) (C) (3, 4) (D) (7, 7)

14. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is :

(A) $(x^2 + y^2)^2 = 4Rx^2y^2$ (B) $(x^2 + y^2)(x + y) = R^2xy$
(C) $(x^2 + y^2)^3 = 4R^2x^2y^2$ (D) $(x^2 + y^2)^2 = 4R^2x^2y^2$

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22. EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad \text{.....(i)}$$

(a) Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms $\mathbf{c_1}$ and $\mathbf{c_2}$ positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

(b) Equation of bisector of acute/obtuse angles :

To find the equation of the bisector of the acute or obtuse angle :

(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if **$\tan\phi < 1$** i.e. $\phi < 45°$ i.e. $2\phi < 90°$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

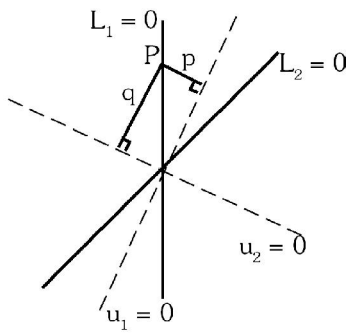
Determine the sign of $\mathbf{a_1a_2 + b_1b_2}$

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use – sign in eq. (1)
–	use – sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

(iii) Another way of identifying an acute and obtuse angle bisector is as follows :



Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x+b_1y+c_1=0$ & $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

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Illustration 31 : For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(iii) bisector of the angle which contains origin.

(iv) bisector of the angle which contains (1, 2).

Solution : Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$$9x - 7y - 41 = 0, \text{ then } \tan\theta = \left|\frac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \frac{11}{3} > 1$$

Hence

(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$

(ii) bisector of the acute angle is $7x + 9y - 3 = 0$

(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1, 2) = 4 \times 1 + 3\times 2 - 6 = 4 > 0$

$L_2(1, 2) = 5 \times 1 + 12 \times 2 + 9 = 38 > 0$

+ve sign will give the required bisector, $\frac{4x+3y-6}{5} = +\frac{5x+12y+9}{13}$

$\Rightarrow$ $9x - 7y - 41 = 0.$

Alternative :

Making c_1 and c_2 positive in the given equation, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle.

Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

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Illustration 32 : A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Solution : Let Q be the point of intersection of the incident ray and the line mirror, then

$$x_1 - 2y_1 - 3 = 0 \quad \& \quad 3x_1 - 2y_1 - 5 = 0$$

on solving these equations, we get

$$x_1 = 1 \quad \& \quad y_1 = -1$$

Since P(–1, –2) be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).

Let P'(h, k) be the image of point P about line mirror, then

$$\frac{h+1}{3} = \frac{k+2}{-2} = \frac{-2(-3+4-5)}{13} \Rightarrow h = \frac{11}{13} \text{ and } k = \frac{-42}{13}.$$

So $\quad P'\left(\frac{11}{13}, \frac{-42}{13}\right)$

Then equation of reflected ray will be

$$(y + 1) = \frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

$$\Rightarrow \quad 2y - 29x + 31 = 0.$$

---

23. FAMILY OF LINES :

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is : $P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

Illustration 33 : Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution : The given family of straight lines can be rewritten as

$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$

or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$

Hence each member of it will pass through a fixed point which is the intersection of

$4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

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Do yourself - 11 :

1. Find the equations of bisectors of the angle between the lines $4x + 3y = 7$ and $24x + 7y - 31 = 0$. Also find which of them is
(a) the bisector of the angle containing origin (b) the bisector of the acute angle.

2. Find the equations of the line which pass through the point of intersection of the lines $4x - 3y = 1$ and $2x - 5y + 3 = 0$ and is equally inclined to the coordinate axes.

3. Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

4. Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true ?
(A) The lines are all parallel. (B) Each line passes through the origin.
(C) The lines are not concurrent (D) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$

5. If a variable line drawn through the intersection of the lines $\frac{x}{3} + \frac{y}{4} = 1$ and $\frac{x}{4} + \frac{y}{3} = 1$, meets the coordinate axes at A and B, $(A \neq B)$, then the locus of the midpoint of AB is :-
(A) $14(x + y)^2 - 97(x + y) + 168 = 0$ (B) $6xy = 7(x + y)$
(C) $7xy = 6(x + y)$ (D) $4(x + y)^2 - 28(x + y) + 49 = 0$

6. The line parallel to x-axis and passing through the point of intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is :
(A) below x-axis at a distance 2/3 from it (B) above x-axis at a distance 3/2 from it
(C) above x-axis at a distance 2/3 from it (D) below x-axis at a distance 3/2 from it

7. The sides of a rhombus ABCD are parallel to the lines, $x - y + 2 = 0$ and $7x - y + 3 = 0$. If the diagonals of the rhombus intersect at P (1,2) and the vertex A (different from the origin) is on the y–axis, then the ordinate of A is :-
(A) 2 (B) $\frac{5}{2}$ (C) $\frac{7}{4}$ (D) $\frac{7}{2}$

8. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three point. The equation of the bisector of the angle PQR is
(A) $\frac{\sqrt{3}}{2}x + y = 0$ (B) $x + \sqrt{3y} = 0$ (C) $\sqrt{3}x + y = 0$ (D) $x + \frac{\sqrt{3}}{2}y = 0$

9. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle $\alpha\left(0 < \alpha < \frac{\pi}{4}\right)$ with the positive direction of x-axis. The equation of its diagonal not passing through the origin is
(A) $y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$ (B) $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$
(C) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$ (D) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$

10. Consider the family of lines $(x - y - 6) + \lambda(2x + y + 3) = 0$ and $(x + 2y - 4) + \lambda'(3x - 2y - 4) = 0$. If the lines of these 2 families are at right angle to each other then find the locus of their point of intersection.

11. Given vertices A (1, 1), B (4, –2) & C (5, 5) of a triangle, find the equation of the perpendicular dropped from C to the interior bisector of the angle A.

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24. PAIR OF STRAIGHT LINES :

(a) Homogeneous equation of second degree :

(i) Let us consider the homogeneous equation of 2nd degree as

$ax^2 + 2hxy + by^2 = 0$ .....(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$$\frac{y}{x} = m \quad \text{(say)}$$

then $a + 2hm + bm^2 = 0$ .....(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

and also, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}\right| = \left|\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}}\right| = \pm\frac{2\sqrt{h^2 - ab}}{a+b}$

These line will be :

(1) **Real and different**, if $h^2 - ab > 0$ (2) **Real and coincident**, if $h^2 - ab = 0$

(3) **Imaginary**, if $h^2 - ab < 0$

(ii) The condition that these lines are :

(1) At **right angles** to each other is **a + b = 0.** i.e. coefficient of $\mathbf{x^2}$ + coefficient of $\mathbf{y^2} = 0$.

(2) **Coincident** is $\mathbf{h^2 = ab}$.

(3) **Equally inclined** to the axes of x is **h = 0**. i.e. coefficient of **xy = 0**.

(iii) Homogeneous equation of 2nd degree $\mathbf{ax^2 + 2hxy + by^2 = 0}$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \equiv y = m_1x \ \& \ y = m_2x \quad \text{and} \quad m_1 + m_2 = -\frac{2h}{b} \ ; \ m_1m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

(iv) Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

(v) The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$

Note : A homogeneous equation of degree n represents n straight lines passing through **origin**.

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(b) The combined equation of angle bisectors :

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, **$a \neq b$, $h \neq 0$**.

Note :

(i) If **$a = b$**, the bisectors are **$x^2 - y^2 = 0$** i.e. **$x - y = 0$, $x + y = 0$**

(ii) If **$h = 0$**, the bisectors are **$xy = 0$** i.e. **$x = 0$, $y = 0$**.

(iii) The two bisectors are always at **right angles**, since we have coefficient of **x^2** + coefficient of **$y^2 = 0$**

(c) General Equation and Homogeneous Equation of Second Degree :

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

(iii) The product of the perpendiculars drawn from the origin to the lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\frac{c}{\sqrt{(a - b)^2 + 4h^2}}\right|$

Illustration 34 : If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : Here $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \quad \Rightarrow 100\lambda = 200$

$\therefore \lambda = 2$ **Ans. (C)**

Do yourself - 12 :

1. Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

2. If the equation $3x^2 + kxy - 10y^2 + 7x + 19y = 6$ represents a pair of lines, find the value of k.

3. If the equation $6x^2 - 11xy - 10y^2 - 19y - 6 = 0$ represents a pair of lines, find their equations. Also find the angle between the two lines.

4. Find the point of intersection and the angle between the lines given by the equation : $2x^2 - 3xy - 2y^2 + 10x + 5y = 0$.

5. The equation $y = \sin x \sin(x + 2) - \sin^2(x+1)$ represents a straight line lying in :
(A) second and third quadrants only (B) third and fourth quadrants only
(C) first, third and fourth quadrants (D) first, second and fourth quadrants

6. If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is
(A) 1 (B) 2 (C) –1/2 (D) –2

7. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals
(A) –3 (B) 1 (C) 3 (D) 1

8. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value
(A) –2 (B) –1 (C) 2 (D) 1

9. If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then
(A) $pq = -1$ (B) $p = q$ (C) $p = -q$ (D) $pq = 1$

10. The pair of lines represented by $3ax^2 + 5xy + (a^2 - 2)y^2 = 0$ are perpendicular to each other for
(A) two values of a (B) $\forall$ a
(C) for one value of a (D) for no values of a

11. If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the y-axis then
(A) $2fgh = bg^2 + ch^2$ (B) $bg^2 \neq ch^2$
(C) $abc = 2fgh$ (D) none of these

12. If the line $y = mx$ bisects the angle between the lines $ax^2 + 2h\,xy + by^2 = 0$ then m is a root of the quadratic equation :
(A) $hx^2 + (a - b)x - h = 0$ (B) $x^2 + h(a - b)x - 1 = 0$
(C) $(a - b)x^2 + hx - (a - b) = 0$ (D) $(a - b)x^2 - hx - (a - b) = 0$

13. The greatest slope along the graph represented by the equation $4x^2 - y^2 + 2y - 1 = 0$, is-
(A) –3 (B) –2 (C) 2 (D) 3

14. Consider a line pair $2x^2 + 3xy - 2y^2 - 10x + 15y - 28 = 0$ and another line L passing through origin with gradient 3. The line pair and line L form a triangle whose vertices are A, B and C.
(a) Find the sum of the cotangents of the interior angles of the triangle ABC.
(b) Find the area of triangle ABC
(c) Find the radius of the circle touching all the 3 sides internally of the triangle.

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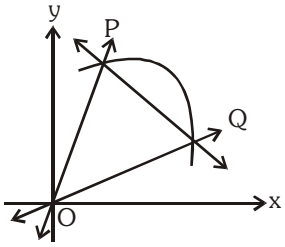
25. EQUATIONS OF LINES JOINING THE POINTS OF INTERSECTION OF A LINE AND A CURVE TO THE ORIGIN :

(a) Let the equation of curve be :



$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

and straight line be

$lx + my + n = 0$ ...(ii)

Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

$\Rightarrow$ $(an^2 + 2gln + cl^2)x^2 + 2(hn^2 + gmn + fln + clm)xy + (bn^2 + 2fmn + cm^2)y^2 = 0$ .....(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second degree curve through the four points of intersection of $f(x, y) = 0$ & $xy = 0$ is given by $f(x, y) + \lambda\, xy = 0$ where $f(x, y) = 0$ is also a second degree curve.

Illustration 35 : The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution : $\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px)+(\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow$ $(4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90°

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore$ $p = \dfrac{-9 \pm \sqrt{81-48}}{8} = \dfrac{-9 \pm \sqrt{33}}{8}$.

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Do yourself - 13 :

1. Find the angle subtended at the origin by the intercept made on the curve

$x^2 - y^2 - xy + 3x - 6y + 18 = 0$ by the line $2x - y = 3$.

2. Find the equation of the lines joining the origin to the points of intersection of the curve

$2x^2 + 3xy - 4x + 1 = 0$ and the line $3x + y = 1$.

3. A straight line is drawn from the point (1, 0) to the curve $x^2 + y^2 + 6x - 10y + 1 = 0$, such that the intercept made on it by the curve subtends a right angle at the origin. Find the equations of the line.

4. Find the equation to the pair of straight lines joining the origin to the intersections of the straight line $y = mx + c$ and the curve $x^2 + y^2 = a^2$. Prove that they are at right angles if $2c^2 = a^2 (1 + m^2)$.

5. Prove that the straight lines joining the origin to the points of intersection of the straight line $kx + hy = 2hk$ with the curve $(x - h)^2 + (y - k)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

6. If line $2x - y - 1 = 0$ intersect the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ in two points A and B, such that $\angle AOB = 90°$, then $\left|\dfrac{a+b+5c}{2g-f}\right|$ is equal to

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